Journal of Statistical Physics, Vol. 124, Nos. 2-4, July/August 2006 (© 2006)
DOI: 10.1007/s10955-005-8087-6

Instabilities in the Chapman-Enskog Expansion and Hyperbolic Burnett Equations

A. V. Bobylev[1]

Received May 11, 2005; accepted August 29, 2005; Published Online: 31 December 2005

It is well-known that the classical Chapman-Enskog procedure does not work at the level of Burnett equations (the next step after the Navier-Stokes equations). Roughly speaking, the reason is that the solutions of higher equations of hydrodynamics (Burnett's, etc.) become unstable with respect to short-wave perturbations. This problem was recently attacked by several authors who proposed different ways to deal with it. We present in this paper one of possible alternatives. First we deduce a criterion for hyperbolicity of Burnett equations for the general molecular model and show that this criterion is not fulfilled in most typical cases. Then we discuss in more detail the problem of truncation of the Chapman-Enskog expansion and show that the way of truncation is not unique. The general idea of changes of coordinates (based on analogy with the theory of dynamical systems) leads finally to nonlinear Hyperbolic Burnett Equations (HBEs) without using any information beyond the classical Burnett equations. It is proved that HBEs satisfy the linearized H-theorem. The linear version of the problem is studied in more detail, the complete Chapman-Enskog expansion is given for the linear case. A simplified proof of the Slemrod identity for Burnett coefficients is also given.

KEY WORDS: Boltzmann equation; Chapman-Enskog method; Burnett equations; hyperbolicity; Perturbation theory; Hydrodynamics.

1. INTRODUCTION

The main objective of this paper is to clarify some aspects of the classical Chapman-Enskog method that bridges the gap between the Boltzmann equation and hydrodynamics. The well-known result of this method is a systematic way of derivation of equations of hydrodynamics having formally any given order of accuracy with respect to the Knudsen number ε (mean free path divided by macroscopic

[1] Department of Mathematics, Karlstad University, 651 88 Karlstad, Sweden; e-mail: alexander-bobylev@kan.se

length). Thus we obtain the sequence: Euler equations, Navier-Stokes equations, Burnett equations, etc.

This is how the Chapman-Enskog method is presented in some old books on classical kinetic theory of gases [8, 16]. On the other hand, Cercignani, in his books, always expressed certain scepticism with respect to higher equations of hydrodynamics, in particular because of uncertainty in boundary conditions [6, 7] (see also [4]). The scepticism was shared by some other authors [10]. Another reason for doubts (not related to boundary conditions) became clear in 1982, when it was proved that Burnett equations for Maxwell molecules are ill-posed [5]. In more physical terms, one can say that any solution of Burnett equations (in particular, the constant equilibrium solution) is unstable with respect to short-wave perturbations (the instability paradox, in terminology of Jin-Slemrod [14]). Attempts to overcome this difficulty were made in last two decades by several authors (see, in particular, papers [11, 18] on the linearized problem and papers [14, 19, 20, 22, 23] on the nonlinear problem). A comprehensive review of different approaches and a complete list of references can be found in [20, 23]. We also mention recent talks by Levermore [17]. All the above approaches are based on a combination of the Chapman-Enskog method with moment methods and on using some higher order in ε terms for regularization of the Burnett equations.

An alternative method considered below does not use any information "beyond the Burnett level" and is based on the following simple idea. Let us consider a general evolution equation for a vector $x(t)$

$$x_t = T(x;\varepsilon) = A(x) + \varepsilon B(x) + \varepsilon^2 C(x) + \dots,$$

where A, B, $C, \dots$ are time-independent differentiable nonlinear operators. In our case $x(t)$ is understood as the vector of hydrodynamical variables, whereas $A(x)$, $B(x)$ and $C(x)$ represent Euler, Navier-Stokes and Burnett terms respectively. A correct remark by Slemrod is that the problem is related not to the Chapman-Enskog expansion itself, but to its truncation at the Burnett level. His approach [14, 19, 20] is, however, quite different from ours.

We consider more carefully the problem of truncation. An obvious observation is that the truncation depends on a choice of coordinates. A formally invertible (for $\varepsilon \to 0$) change of variables

$$y = x + \varepsilon^2 R(x) \Rightarrow x = y - \varepsilon^2 R(y) + \dots,$$

with an arbitrary time-independent differentiable operator R, leads to the equation

$$y_t = \widetilde{T}(y;\varepsilon) = A(y) + \varepsilon B(y) + \varepsilon^2 \widetilde{C}(y) + \dots, \widetilde{C}(y) = C(y) + R'_y A(y) - A'_y R(y),$$

where R'_y and A'_y are the Fréchet derivatives (linear operators) of $R(y)$ and $A(y)$ respectively. Thus, the result of truncation at the level $O(\varepsilon^2)$ depends, generally speaking, on an arbitrary operator R (or, equivalently, on a choice of coordinates). The same, of course, can be done at any order $O(\varepsilon^n)$, $n \geq 1$. Note that even the classical Navier-Stokes equations for $n = 1$ are not uniquely defined in this sense. A remaining non-trivial problem is to choose new coordinates (the operator R) in the most reasonable way. Similar ideas are widely used in the theory of dynamical systems, see, for example, the Poincaré method of normal forms [1].

The paper is organized as follows. In Section 2 we transform the Boltzmann equation in such a way that makes the Navier-Stokes equations almost obvious and clearly indicates "what is missing in Navier-Stokes equations" as compared to the Boltzmann equation. Then the Burnett equations arise quite naturally as the first correction to Navier-Stokes equations in Section 3 We consider these equations under very general assumptions on intermolecular forces and derive a criterion for hyperbolicity of Burnett equations. The criterion roughly means that they are hyperbolic in a very narrow interval $1 \leq \mathrm{Pr} \leq 5/4$ of Prandtl numbers (in contrast to the realistic value $\mathrm{Pr} \approx 2/3$). In order to clarify a mathematical reason for the instability paradox, we consider in Sections 4,5 the linearized problem. Then the idea of change of variables arises quite naturally and we consider in Section 6 an appropriate family of transformations of equations of hydrodynamics. This leads to hyperbolic Burnett equations (HBEs) that satisfy a linearized version of H-theorem (Section 7). The hyperbolic Burnett equations are discussed in detail in Section 8 This section contains all necessary information for practical use of HBEs. As a by-product result, a simplified proof of the Slemrod identity [21] is given at the end of Section 8 The complete Chapman-Enskog expansion for a broad class of linear equations (the linearized Boltzmann equation belongs to this class) is described in Appendix.

The reader who is not interested in mathematical aspects of the problem can skip Sections 4,5 and proceed directly to Section 6. On the other hand, Sections 4,5 and Appendix, based on classical perturbation theory for linear operators [15], are important for clarifying the mathematical nature of the instability paradox (loss of symmetry for approximate linear evolution operators). The general structure of the complete Chapman-Enskog expansion (Appendix) makes the above sketched change of coordinates quite natural from mathematical point of view.

We do not know yet whether or not any nonlinear version of H-theorem holds for HBEs (such versions are proved for some other methods of regularization of Burnett equations [14, 21]). A connection of HBEs with thermodynamics is another interesting open problem.

In order to avoid a misunderstanding we note that the above discussed difficulties with the Chapman-Enskog method are typical for the Euler limit. An alternative scaling that leads to the incompressible Navier-Stokes equations [3, 9], seems to be in complete agreement with corresponding Chapman-Enskog expansion [13].

2. TRANSFORMATION OF THE BOLTZMANN EQUATION

We consider the Boltzmann equation [7]

$$\mathcal{D}f = \frac{1}{\varepsilon} Q(f, f), \mathcal{D} = \partial_t + v \cdot \partial_x, \tag{1}$$

for the distribution function $f(x, v, t; \varepsilon)$, where the variables $x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$ and $t > 0$ correspond respectively to position, velocity and time; the parameter $\varepsilon > 0$ denotes the Knudsen number. The Boltzmann collision operator reads

$$Q(f, f) = \int\limits_{\mathbb{R}^3 \times \mathbb{S}^2} dw d\omega \, |u| \, \sigma \left(|u|, \frac{u \cdot \omega}{|u|} \right) \left[f(v') f(w') - f(v) f(w) \right],$$

$$u = v - w, \omega \in \mathbb{S}^2, v' = \frac{1}{2}(v + w + |u| \, \omega), w' = \frac{1}{2}(v + w - |u| \, \omega), \tag{2}$$

where $\sigma(|u|, \cos\theta)$ is the differential cross-section that corresponds to the scattering angle $\theta \in [0, \pi]$ (irrelevant arguments x, t and ε of the function f are omitted in Eq. (2)).

We denote for brevity

$$\langle f, g \rangle = \langle fg \rangle = \int\limits_{\mathbb{R}^3} dv f(v) g(v) \tag{3}$$

and introduce the so-called hydrodynamic variables (the density ρ, the bulk velocity $u \in \mathbb{R}^3$ and the temperature T)

$$\rho = \langle f \rangle, \rho u = \langle f, v \rangle, \rho T = \frac{1}{3} \left\langle f, |c|^2 \right\rangle, c = v - u. \tag{4}$$

We shall use below just a few basic properties of the collision integral:

(A) $\langle \Psi, Q(f, f) \rangle = 0$, for any $f(v)$ if and only if $\Psi = \Psi(v) \in Span\left(1, v, |v|^2\right)$;
(B) $Q(f, f) = 0$, $f \geq 0$, if and only if $f = \exp\left(\alpha + \beta \cdot v - \gamma |v|^2\right)$, where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^3$, $\gamma \in \mathbb{R}_+$ are any constant parameters.

All considerations in this paper are quite formal and therefore we assume in advance that the cross-section $\sigma(|u|, \cos\theta)$ in Eq. (2) and the solution f of Eq. (1) satisfy all necessary restrictions such that all integrals are convergent, etc. In order to describe an asymptotic behavior of $f(x, v, t; \varepsilon)$ for small positive ε we introduce the Maxwellian distribution

$$M = (2\pi T)^{-3/2} \exp\left(-\frac{|c|^2}{2T}\right), c = v - u, \tag{5}$$

and represent the solution of Eq. (1) as a sum

$$f = \rho M + \varepsilon F, \tag{6}$$

where $\rho(x, t; \varepsilon)$, $u(x, t; \varepsilon)$, $T(x, t; \varepsilon)$, correspond to the "true" hydrodynamic moments of f. Hence,

$$\langle \Psi, F \rangle = 0, \langle \Psi, \mathcal{D}\rho M \rangle + \epsilon \langle \Psi, \mathcal{D}F \rangle = 0, \tag{7}$$

for any $\Psi \in Span(1, v, |v|^2 right)$. This leads to the usual set of hydrodynamic conservation laws

$$\rho_t + div\, \rho u = 0, \frac{\partial}{\partial t}\rho u_\alpha + \frac{\partial}{\partial x_\beta}\left(\rho u_\alpha u_\beta + p\delta_{\alpha\beta} + \varepsilon\pi_{\alpha\beta}\right) = 0,$$

$$\frac{\partial}{\partial t}\rho\left(|u|^2 + 3T\right) + div\, \rho u\left(|u|^2 + 5T\right) + 2\varepsilon\frac{\partial}{\partial x_\alpha}\left(\pi_{\alpha\beta}u_\beta + q_\alpha\right) = 0, \tag{8}$$

where

$$p = \rho T, \pi_{\alpha\beta} = \left\langle F, c_\alpha c_\beta \right\rangle, q_\alpha = \frac{1}{2}\left\langle c_\alpha |c|^2, F\right\rangle, \alpha, \beta = 1, 2, 3. \tag{9}$$

The usual rule of summation over repeating indexes is assumed here and below.

The equation for $F(x, v, t; \varepsilon)$ reads

$$\mathcal{D}\rho M + \varepsilon\mathcal{D}F = \rho M L\left(F/M\right) + \varepsilon Q\left(F, F\right), \tag{10}$$

where the linearized collision operator L is defined by the equality

$$Q\left(M, Mg\right) + Q\left(Mg, M\right) = MLg. \tag{11}$$

We denote by **H** the Hilbert space with the scalar product

$$\langle g_1, g_2\rangle_M = \langle Mg_1, g_2\rangle \tag{12}$$

and consider L as an operator acting from **H** to **H**. We shall use the following basic properties of L (in addition to properties (A) and (B) of the nonlinear operator Q):

(C) $Lg = 0$ if and only if $g \in \mathbf{N}(L) = Span\left(1, v, |v|^2\right)$;
(D) L is self-adjoint and semi-negative operator, i.e.

$$\langle g_1, Lg_2\rangle_M = \langle Lg_1, g_2\rangle_M, \langle g, Lg\rangle_M \le 0; \tag{13}$$

(E) $\mathbf{H} = \mathbf{N}(L) \oplus \mathbf{R}(L)$, where $\mathbf{R}(L) = L\mathbf{H}$ is a range of the operator L in **H**, $\oplus$ denotes the orthogonal sum (with respect to the scalar product (12)).

Then the problem

$$Lg = \phi, g \in \mathbf{R}(L), \phi \in \mathbf{R}(L) \tag{14}$$

has a unique solution $g = L^{-1}\phi$. We extend the operator $L^{-1} : \mathbf{R}(L) \to \mathbf{R}(L)$ to the whole $\mathbf{H}$ by introducing the linear operator $\widetilde{L}^{-1} : \mathbf{H} \to \mathbf{R}(L)$ such that

$$\widetilde{L}^{-1}\phi = \begin{cases} L^{-1}\phi, & \text{if} \phi \in \mathbf{R}(L) \\ 0, & \text{if} \phi \in \mathbf{N}(L). \end{cases} \tag{15}$$

Then Eq. (10) can be transformed to

$$F = M\left[\widetilde{L}^{-1}(\mathcal{D} \ln \rho M) + \frac{\varepsilon}{\rho}\widetilde{L}^{-1}\frac{\mathcal{D}F - Q(F,F)}{M}\right]. \tag{16}$$

We denote

$$\widetilde{Q}(F) = Q(F,F),\ F_0 = M\widetilde{L}^{-1}(\mathcal{D} \ln \rho M) = -\,M\widetilde{L}^{-1}\left(c_\alpha \cdot \frac{\partial}{\partial x_\alpha}\frac{|c|^2}{2T}\right),\ c = v - u, \tag{17}$$

the same notation $c = v - u$ for the thermal velocity is often used below. Then, omitting tildes, we obtain

$$F = F_0 + \frac{\varepsilon}{\rho} M L^{-1} \frac{\mathcal{D}F - Q(F)}{M}, \tag{18}$$

where

$$F_0 = M\left[\frac{1}{T}\frac{\partial u_\alpha}{\partial x_\beta}\phi_{\alpha\beta}(c) + \frac{1}{T^2}\frac{\partial T}{\partial x_\alpha}\phi_\alpha(c)\right],$$

$$\phi_{\alpha\beta} = L^{-1}\left(c_\alpha c_\beta - \frac{|c|^2}{3}\delta_{\alpha\beta}\right),\ \phi_\alpha = L^{-1}\frac{c_\alpha}{2}(|c|^2 - 5T). \tag{19}$$

The general equation of hydrodynamics (the second equation (7) with any $\Psi \in Span\left(1, v, |v|^2\right)$ independent of x and t) reads now

$$\langle \Psi, \mathcal{D}\rho M\rangle + \varepsilon \frac{\partial}{\partial x_\alpha}\langle c_\alpha \Psi, F_0\rangle + \varepsilon^2 \frac{\partial}{\partial x_\alpha}\frac{1}{\rho}\left\langle M c_\alpha \Psi, L^{-1}\frac{\mathcal{D}F - Q(F)}{M}\right\rangle = 0. \tag{20}$$

The operator L^{-1} (15) is obviously self-adjoint in $\mathbf{H}$ and therefore we obtain

$$\langle \Psi, \mathcal{D}\rho M\rangle + \varepsilon \frac{\partial}{\partial x_\alpha}\left\langle \Phi_\alpha(\Psi), c \cdot \frac{\partial M}{\partial x}\right\rangle + \varepsilon^2 \frac{\partial}{\partial x_\alpha}\frac{1}{\rho}\langle \Phi_\alpha(\Psi), \mathcal{D}F - Q(F)\rangle = 0,$$

$$\Phi_\alpha(\Psi) = L^{-1}(c_\alpha \Psi);\Psi = 1, v, |v|^2, \tag{21}$$

where F satisfies Eq. (18). The equations (21) are obviously exact. If we neglect the third term having the order $O(\varepsilon^2)$, then Eqs. (21) reduce to the classical Navier-Stokes equations. Thus, the Navier-Stokes equations arise quite naturally (also for the stationary Boltzmann equation with $\mathcal{D} = v \cdot \partial_x$). The difficulties begin with the attempt to solve Eq. (18) for small positive ε and then to use the equations

(21) with terms of order $O(\varepsilon^2)$ (Burnett equations). We consider this problem in the next Section 3.

3. BURNETT EQUATIONS

The equation (18) can be written as

$$F = F_0 + \varepsilon F_1 + O(\varepsilon^2),\, F_1 = \frac{1}{\rho} M L^{-1} \frac{\mathcal{D}F_0 - Q\,(F_0)}{M}, \tag{22}$$

where F_0 is given in Eq. (19). The Burnett equations are the equations (8) with

$$\begin{aligned} \pi_{\alpha\beta} &= \pi_{\alpha\beta}^{NS} + \varepsilon\pi_{\alpha\beta}^{B},\, q_\alpha = q_\alpha^{NS} + \varepsilon q_\alpha^{B}, \\ \pi_{\alpha\beta}^{NS} &= \left\langle F_0, c_\alpha c_\beta \right\rangle, q_\alpha^{NS} = \frac{1}{2}\left\langle F_0, c_\alpha\, |c|^2 \right\rangle, \\ \pi_{\alpha\beta}^{B} &= \left\langle F_1, c_\alpha c_\beta \right\rangle, q_\alpha^{B} = \frac{1}{2}\left\langle F_1, c_\alpha\, |c|^2 \right\rangle, \end{aligned} \tag{23}$$

plus the rule of calculation of $\mathcal{D}F_0$ in Eq. (22) (see below). We can represent the Burnett terms in the equivalent form:

$$\pi_{\alpha\beta}^{B} = \frac{1}{\rho}\left\langle \phi_{\alpha\beta}, \mathcal{D}F_0 - Q\,(F_0) \right\rangle, q_\alpha^{B} = \frac{1}{\rho}\left\langle \phi_\alpha, \mathcal{D}F_0 - Q\,(F_0) \right\rangle \tag{24}$$

in the notations (19). We note that

$$\begin{aligned} \mathcal{D}_0\rho &= -\rho\, div\, u, \mathcal{D}_0 u = -\frac{1}{\rho}\nabla p + O(\varepsilon), \\ \mathcal{D}_0 T &= -\frac{2}{3} T\, div\, u + O(\varepsilon), \mathcal{D}_0 = \partial_t + u\cdot\partial_x, \end{aligned} \tag{25}$$

and this explains the rule of calculation of the derivative

$$\mathcal{D}F_0 = \mathcal{D}_0 F_0 + c\cdot\frac{\partial F_0}{\partial x}, \mathcal{D}_0 = \partial_t + u\cdot\partial_x, \tag{26}$$

in Eqs. (22), (24): we should use the Euler formulas (25) and neglect all terms $O(\varepsilon)$. This completes the definition of the Burnett equations.

One can easily verify that Eqs. (24) can be transformed to the following form:

$$\begin{aligned} \pi_{\alpha\beta}^{B} &= \mathcal{D}_0 P_{\alpha\beta} + \frac{1}{\rho}\left[\frac{\partial}{\partial x_\gamma}\left\langle c_\gamma\phi_{\alpha\beta}, F_0\right\rangle - \Delta_{\alpha\beta}\right], \\ q_\alpha^{B} &= \mathcal{D}_0 Q_\alpha + \frac{1}{\rho}\left[\frac{\partial}{\partial x_\beta}\left\langle c_\beta\phi_\alpha, F_0\right\rangle - \Delta_\alpha\right], \end{aligned} \tag{27}$$

where

$$P_{\alpha\beta} = \frac{1}{\rho}\left\langle F_0, \phi_{\alpha\beta}\right\rangle, \; \Delta_\alpha = \left\langle F_0, \mathcal{D}\phi_\alpha\right\rangle + \left\langle \phi_\alpha, Q\left(F_0\right)\right\rangle,$$

$$Q_\alpha = \frac{1}{\rho}\left\langle F_0, \phi_\alpha\right\rangle, \; \Delta_{\alpha\beta} = \left\langle F_0, \mathcal{D}\phi_{\alpha\beta}\right\rangle + \left\langle \phi_{\alpha\beta}, Q\left(F_0\right)\right\rangle. \tag{28}$$

Eqs. (27) are more convenient for our goals, since the reminder terms $\Delta_{\alpha\beta}$ and Δ_α are bilinear forms with respect to first derivatives of ρ, T and u. Therefore they disappear in the linearized (near the constant solution) equation and do not influence higher derivatives in the nonlinear equations. Therefore we concentrate on the "main" terms in Eqs. (27).

The straightforward calculation of the integrals with F_0 given in Eq. (19) yields

$$P_{\alpha\beta} = \frac{A}{\rho}\overline{\frac{\partial u_\alpha}{\partial x_\beta}}, \; Q_\alpha = \frac{B}{\rho}\frac{\partial T}{\partial x_\alpha}, \; \frac{\partial}{\partial x_\gamma}\left\langle c_\gamma \phi_{\alpha\beta}, F_0\right\rangle = \overline{\frac{\partial}{\partial x_\alpha} C \frac{\partial T}{\partial x_\beta}},$$

$$\frac{\partial}{\partial x_\beta}\left\langle c_\beta \phi_\alpha, F_0\right\rangle = \frac{\partial}{\partial x_\beta} C T \overline{\frac{\partial u_\alpha}{\partial x_\beta}}, \tag{29}$$

where

$$\overline{a_{\alpha\beta}} = \frac{1}{2}\left(a_{\alpha\beta} + a_{\beta\alpha} - \frac{2}{3}\delta_{\alpha\beta} Tr\; a\right), \; Tr\; a = a_{11} + a_{22} + a_{33},$$

$$A = A\left(T\right) = \frac{1}{5T}\left\langle \phi_{\alpha\beta}, \phi_{\alpha\beta}\right\rangle_M, \; B = B\left(T\right) = \frac{1}{3T^2}\left\langle \phi_\alpha, \phi_\alpha\right\rangle_M,$$

$$C = C\left(T\right) = \frac{1}{5T^2}\left\langle c_\alpha \phi_\beta, \phi_{\alpha\beta}\right\rangle_M. \tag{30}$$

Then we use the identity

$$\mathcal{D}_0 \frac{\partial}{\partial x_\beta} = \frac{\partial}{\partial x_\beta}\mathcal{D}_0 - \frac{\partial u_\gamma}{\partial x_\beta}\frac{\partial}{\partial x_\gamma} \tag{31}$$

and the above described rule for calculating $\mathcal{D}_0\rho$, $\mathcal{D}_0 u$, $\mathcal{D}_0 T$ by Eqs. (25). Thus we obtain

$$\mathcal{D}_0 P_{\alpha\beta} = -\frac{A}{\rho}\left[\overline{\frac{\partial}{\partial x_\alpha}\frac{1}{\rho}\frac{\partial p}{\partial x_\beta}} + \overline{\frac{\partial u_\alpha}{\partial x_\gamma}\frac{\partial u_\gamma}{\partial x_\beta}} - \alpha\left(T\right)\left(div\, u\right)\overline{\frac{\partial u_\alpha}{\partial x_\beta}}\right],$$

$$\mathcal{D}_0 Q_\alpha = -\frac{B}{\rho}\left[\frac{2}{3}\frac{\partial}{\partial x_\alpha} T\, div\, u + \frac{\partial u_\beta}{\partial x_\alpha}\frac{\partial T}{\partial x_\beta} - \beta\left(T\right)\left(div\, u\right)\frac{\partial T}{\partial x_\alpha}\right],$$

$$\alpha\left(T\right) = 1 - \frac{2}{3}\frac{T A'\left(T\right)}{A\left(T\right)}, \; \beta\left(T\right) = 1 - \frac{2}{3}\frac{T B'\left(T\right)}{B\left(T\right)}. \tag{32}$$

Hence,

$$\pi^B_{\alpha\beta} = -\frac{1}{\rho}\left[\overline{\Pi^{(1)}_{\alpha\beta}} + \overline{\Pi^{(2)}_{\alpha\beta}} + \Delta_{\alpha\beta}\right], q^B_\alpha = -\frac{1}{\rho}\left[S^{(1)}_\alpha + S^{(2)}_\alpha + \Delta_\alpha\right], \tag{33}$$

where

$$\begin{aligned}
\Pi^{(1)}_{\alpha\beta} &= A\frac{\partial}{\partial x_\alpha}\frac{1}{\rho}\frac{\partial p}{\partial x_\beta} - \frac{\partial}{\partial x_\alpha}C\frac{\partial T}{\partial x_\beta}, \; p = \rho T,\\
\Pi^{(2)}_{\alpha\beta} &= A\left[\frac{\partial u_\alpha}{\partial x_\gamma}\frac{\partial u_\gamma}{\partial x_\beta} - \alpha(T)\frac{\partial u_\alpha}{\partial x_\beta} divu\right],\\
S^{(1)}_\alpha &= \frac{2B}{3}\frac{\partial}{\partial x_\alpha}T divu - \frac{\partial}{\partial x_\beta}CT\overline{\frac{\partial u_\alpha}{\partial x_\beta}},\\
S^{(2)}_\alpha &= B\frac{\partial T}{\partial x_\beta}\left[\frac{\partial u_\beta}{\partial x_\alpha} - \beta(T)\delta_{\alpha\beta} divu\right],
\end{aligned} \tag{34}$$

other notations are given in Eqs. (28), (30), (32).

These formulas are sufficient to explain why Burnett equations are ill-posed. Considering just the terms with higher derivatives we transform Eqs. (8), (23), (33) to

$$\rho_t = \ldots, \frac{\partial u_\alpha}{\partial t} = \frac{\varepsilon^2}{\rho^2}\frac{\partial\overline{\Pi^{(1)}_{\alpha\beta}}}{\partial x_\beta} + \ldots, T_t = \frac{2}{3}\frac{\varepsilon^2}{\rho^2} divS^{(1)} + \ldots, \tag{35}$$

where dots denote terms that do not contain third derivatives. Then, after simple calculations, we obtain

$$\begin{aligned}
\rho_t &= \ldots, u_t = \frac{2}{3}\frac{\varepsilon^2}{\rho^2}\left[\frac{AT}{\rho}\Delta(\nabla\rho) + (A-C)\Delta(\nabla T)\right] + \ldots,\\
T_t &= \left(\frac{2}{3}\right)^2\frac{\varepsilon^2}{\rho^2}T(B-C)\Delta divu.
\end{aligned} \tag{36}$$

It is sufficient to consider 1d solutions

$$\rho = \rho(x_1, t), u = \{u_1(x_1, t), 0, 0\}, T = T(x_1, t),$$

then the matrix M of the coefficients for third derivatives reads

$$\begin{pmatrix} 0 & 0 & 0 \\ \frac{AT}{\rho} & 0 & A-C \\ 0 & \frac{2(B-C)T}{3} & 0 \end{pmatrix}. \tag{37}$$

Its non-zero eigenvalues are

$$\lambda_\pm = \pm\left[\frac{2}{3}T(B-C)(A-C)\right]^{1/2}.$$

Hence, the hyperbolicity condition (under obvious assumption $T > 0$) reads

$$(B - C)(A - C) \geq 0. \tag{38}$$

It is easy to verify that this condition is not fulfilled in most typical molecular models for the Boltzmann equation. In order to do this we use temporary notations

$$\Psi_{\alpha\beta}(c) = c_\alpha c_\beta - \frac{|c|^2}{3}\delta_{\alpha\beta},\ \Psi_\alpha(c) = \frac{c_\alpha}{2}\left(|c|^2 - 5T\right) \tag{39}$$

and represent the Navier-Stokes terms in Eqs. (23) by equalities

$$\pi_{\alpha\beta}^{NS} = -2\mu(T)\overline{\frac{\partial u_\alpha}{\partial x_\beta}},\ q_\alpha^{NS} = -\lambda(T)\frac{\partial T}{\partial x_\alpha}, \tag{40}$$

where $\mu(T)$ and $\lambda(T)$ denote respectively the coefficients of viscosity and heat conductivity. It follows from Eq. (19) that

$$\mu(T) = -\frac{1}{10T}\left\langle\phi_{\alpha\beta}, \Psi_{\alpha\beta}\right\rangle_M,\ \lambda(T) = -\frac{1}{3T^2}\left\langle\phi_\alpha, \Psi_\alpha\right\rangle_M. \tag{41}$$

The usual approximation [8] for functions $\phi_{\alpha\beta}(c)$ and $\phi_\alpha(c)$ (19) is given by

$$\phi_{\alpha\beta} \approx a(T)\Psi_{\alpha\beta}(c),\ \phi_\alpha \approx b(T)\Psi_\alpha(c), \tag{42}$$

then

$$a(T) = -\frac{\mu(T)}{T},\ b(T) = -\frac{2\lambda(T)}{5T} \tag{43}$$

since

$$\left\langle\Psi_{\alpha\beta}, \Psi_{\alpha\beta}\right\rangle_M = 10T^2,\ \left\langle\Psi_\alpha, \Psi_\alpha\right\rangle_M = \frac{15}{2}T^3. \tag{44}$$

The approximation (42) is exact for Maxwell molecules, moreover $\lambda(T) = 15\mu(T)/4$ in that case. By using Eqs. (42), (43) we evaluate the integrals (30) and obtain

$$A = 2\frac{\mu^2}{T},\ B = \frac{2\lambda^2}{5T},\ C = \frac{4\lambda\mu}{5T}. \tag{45}$$

Therefore

$$A - C = \frac{4\lambda\mu}{5T}\left(\frac{5\mu}{2\lambda} - 1\right),\ B - C = \frac{2\lambda^2}{5T}\left(1 - \frac{\mu}{\lambda}\right).$$

The ratio

$$Pr = \frac{5}{2}\frac{\mu(T)}{\lambda(T)} \tag{46}$$

is called in fluid mechanics the Prandtl number (for monoatomic gases) [16]. It is well known that the approximate equality $Pr \simeq 2/3$ (exact for Maxwell molecules) holds for all typical molecular models (hard spheres, etc.). On the other hand, the hyperbolicity condition (38) can be approximately (under the assumption (42)) written as

$$1 \leq \text{Pr} \leq 5/4. \tag{47}$$

The realistic value $Pr = 2/3$ obviously violates this condition. Therefore the Burnett equations are probably ill-posed for all typical molecular models, though our proof is rigorous just for Maxwell molecules (all above formulas are exact in this case). We note that $Pr = 1$ for BGK model, however this model is too unrealistic.

In the next section we consider a simplified (linear) version of the general problem in order to understand why such irregularities appear in asymptotic expansions. Then it will be quite clear how to remove the irregularities.

The reader, who is interested just in fluid mechanics applications, can skip Sections 4 and 5, and proceed directly to Section 6.

4. LINEARIZED BOLTZMANN EQUATION AND INSTABILITIES IN ASYMPTOTIC EXPANSIONS

We consider the Boltzmann equation (1) and linearize the equation near the standard absolute Maxwellian:

$$f = M_0 + M_0^{1/2} g,\ M_0 = (2\pi)^{-3/2} \exp\left(-\frac{|v|^2}{2}\right). \tag{48}$$

Neglecting the nonlinear on g terms we obtain

$$g_t + v \cdot g_x = \frac{1}{\varepsilon} K g,\ Kg = M_0^{-1/2}\big[Q(M_0, M_0^{1/2} g) + Q(M_0^{1/2} g, M_0)\big]. \tag{49}$$

It is well-known that the same Chapman-Enskog procedure can be used for Eq. (49) (with obvious modifications) and it leads again to ill-posed linearized Burnett equations [5]. The linear problem (49) is, however, much simpler and it can be studied in detail. The following properties of K are important (see properties (C), (D), (E) of the operator L in Section 2):

(C′) $\mathbf{N}(K) = Span(M_0^{1/2}, M_0^{1/2} v, M_0^{1/2} |v|^2)$;
(D′) $\langle g_1, K g_2 \rangle = \langle K g_1, g_2 \rangle,\ \langle g, K g \rangle \leq 0$;
(E′) $\mathbf{N}(K) \oplus \mathbf{R}(K) = \mathbf{L}_2(\mathbb{R}^3)$,

where the scalar product is defined in Eq. (3) (the only reason to consider K, instead of L, is to avoid the Maxwellian weight in the scalar product).

Then we make the Fourier transform

$$\widehat{g}(k, v, t) = \int_{\mathbb{R}^3} dx\; g(x, v, t)e^{-ik\cdot x} \tag{50}$$

and obtain

$$\widehat{g}_t + ik \cdot v\widehat{g} = \frac{1}{\varepsilon}K\widehat{g}. \tag{51}$$

It is convenient to forget about the Boltzmann equation for a while and to study a more general problem related to Eq. (51).

Remark 1: The rest of Section 4 and Section 5 do not use any information about the Boltzmann equation. Therefore we shall use in this part of the paper the same letters $A, B, P, Q, x, u,$ etc. without any connection with notations of Sections 2,3. Hopefully this will not cause any confusion for the reader.

Let $\mathbf{E}$ be a unitary space with the usual (complex) scalar product $(\cdot, \cdot)$. We use below notations $x, y, u, \ldots$ for vectors of $\mathbf{E}$ and denote by capital letters $A, B, C, \ldots$ linear operators. Subsets of $\mathbf{E}$ are denoted by bold capital letters $\mathbf{N}, \mathbf{M}, \ldots$. All our considerations can be justified rigorously if $\dim \mathbf{E} < \infty$ [15], though they formally remain the same in the case $\dim \mathbf{E} = \infty$.

We consider the Cauchy problem for a vector $u(t) \in \mathbf{E}$, $t \geq 0$:

$$u_t + iBu + \frac{1}{\varepsilon}Au = 0, \varepsilon > 0,\ u|_{t=0} = u_0, \tag{52}$$

under the following assumptions about the operators A and B:

[i] both A and B are real and symmetric, i.e. $\overline{A} = A$, $\overline{B} = B$, $(u_1, Au_2) = (Au_1, u_2)$, $(u_1, Bu_2) = (Bu_1, u_2)$;

[ii] A is semi-positive, i.e. $(u, Au) \geq 0$ for any $u \in \mathbf{E}$;

[iii] the equation $Au = 0$ has precisely $1 \leq m \leq \dim \mathbf{E}$ linearly independent solutions $u = e_\alpha$, $\alpha = 1, \ldots, m$, then

$$\mathbf{N}(A) = Ker\, A = Span(e_1, \ldots, e_m). \tag{53}$$

The image of A is denoted by $\mathbf{R}(A) = A\mathbf{E}$. We assume that $\mathbf{E} = \mathbf{N}(A) \oplus \mathbf{R}(A)$, this assumption is always fulfilled if $\dim \mathbf{E} < \infty$. Thus, the linearized Boltzmann equation (51) is a particular case of Eq. (52).

The conditions [i], [ii] lead to the following identity

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = -\frac{1}{\varepsilon}(Au, u) \leq 0,\ \|u\|^2 = (u, u), \tag{54}$$

i.e. the trivial solution $u = 0$ is the stable node for Eq. (52). Our goal in this section is to understand why this property does not hold for the truncated Chapman-Enskog expansion.

We denote

$$\mathbf{X} = \mathbf{N}(A), \mathbf{Y} = \mathbf{R}(A) = \mathbf{X}^{\perp}, \dim \mathbf{X} = m, \tag{55}$$

and introduce an orthogonal projector $P : \mathbf{E} \to \mathbf{X}$ onto the subspace $\mathbf{X}$.

Then

$$PA = AP = 0, P^2 = P. \tag{56}$$

The Chapman-Enskog method for Eq. (52) implies a decomposition

$$u = x + \varepsilon y, x = Pu \in \mathbf{X}, \varepsilon y = (1 - P)u \in \mathbf{Y},$$

that leads to coupled equations

$$x_t + iPB(x + \varepsilon y) = 0, \varepsilon y_t + i(1 - P)B(x + \varepsilon y) + Ay = 0. \tag{57}$$

We denote by $y = A^{-1}z$ a unique solution of the problem

$$Ay = z, y \in \mathbf{Y}, z \in \mathbf{Y},$$

and extend A^{-1} to the whole space $\mathbf{E}$ by equality

$$U = A^{-1}(1 - P). \tag{58}$$

Then the second equation (57) is equivalent to

$$y = -iUBx - \varepsilon U(y_t + iBy)$$

and therefore we obtain

$$x_t + iPBx + \varepsilon PBUBx = i\varepsilon^2 PBU(y_t + iBy). \tag{59}$$

We note that

$$y = -iUBx - O(\varepsilon), y_t = -iUBx_t + O(\varepsilon) = -UBPBx + O(\varepsilon), \varepsilon \to 0.$$

This leads to a sequence of truncated "equations of hydrodynamics:"

$$(E)\ x_t + iPBx = 0; \ \ (N - S)\ x_t + iPBx + \varepsilon PBUBx = 0;$$

$$(B)\ x_t + iPBx + \varepsilon PBUBx = i\varepsilon^2 PBU(BU - UBP)Bx \tag{60}$$

where the notations (E), $(N - S)$, (B) correspond respectively to Euler, Navier-Stokes and Burnett approximations.

We can now easily see what happens at the "Burnett level" of approximation. Note that all operators U, B and P are real and symmetric. Since $x \in \mathbf{X}$, we have the identity $Px = x$ and therefore $PBx = PBPx$, etc.. Then each of the three equations (60) can be written as

$$x_t + i\widetilde{B}(\varepsilon)x + \varepsilon\widetilde{A}x = 0, \varepsilon > 0, x \in \mathbf{X}, \tag{61}$$

where $\widetilde{A}$ and $\widetilde{B}$ are operators acting from $\mathbf{X}$ to $\mathbf{X}$. The operators $\widetilde{B}(0) = PBP$ and $\widetilde{A} = PBUBP$ are obviously symmetric, moreover $(\widetilde{A}x, x) \geq 0$. Therefore the identity similar to Eq. (54) leads to the equality $\|x(t)\| = \|x(0)\|$ for Euler equations and to the inequality $\|x(t)\| \leq \|x(0)\|$ for Navier-Stokes equations. However, nothing like that can be proved for Burnett equations since $\widetilde{B}(\varepsilon)$ is not symmetric for $\varepsilon > 0$.

The loss of symmetry is the real reason why the Burnett equations (related to the Boltzmann equation (49)) are ill-posed. The matter is that the operator $B = k \cdot v$ in Eq. (51) is proportional to $|k|$ and the loss of symmetry of the operator $\widetilde{B}(\varepsilon)$ in the Burnett equation becomes crucial when $|k| \to \infty$.

The complete Chapman-Enskog expansion for Eq. (52) and its regularization are discussed in detail in Appendix. This material, however, is not necessary if we just want to understand what can be done at the Burnett level of truncation. We consider this question in Section 6

5. REGULARIZATION

The general equation for $x(t)$ obtained by the Chapman-Enskog method reads (see Eqs. (60) , (61))

$$x_t + i(B_0 + \varepsilon^2 B_1)x + \varepsilon A_0 x + \ldots = 0, \tag{62}$$

where dots denote terms of order $O(\varepsilon^n)$, $n \geq 3$,

$$B_0 = PBP, A_0 = PBUBP, B_1 = PB(U^2BP - UBU)BP \tag{63}$$

If we neglect terms of order $O(\varepsilon^n)$, $n \geq 3$, then we obtain Burnett equations. This is the simplest way of truncation of Eq. (62), but not the only one. Let us consider a more general way of truncation.

We substitute $x \in \mathbf{X}$ in Eq. (62) by another variable $z \in \mathbf{X}$ such that

$$z = x + \varepsilon^2 Rx \Rightarrow x = z - \varepsilon^2 Rz + \ldots, \tag{64}$$

where $R : \mathbf{X} \to \mathbf{X}$ is a time-independent linear operator, dots denote terms of order $O(\varepsilon^3)$. Then we obtain the following equation for $z(t)$:

$$z_t + i(B_0 + \varepsilon^2 \widetilde{B}_1)z + \varepsilon A_0 z + \ldots = 0, \widetilde{B}_1 = B_1 + (RB_0 - B_0 R). \tag{65}$$

Neglecting terms of order $O(\varepsilon^3)$, we obtain a family of "Burnett equations" that depends on arbitrary operator R. Each member of this family has formally the same order of approximation as the usual Burnett equations with $R = 0$. It is now easy to find the operator R that "kills" above discussed instabilities. The reason for instabilities is that B_1 in Eq. (62) is not symmetric. Hence, we need to choose R in such a way that

$$\widetilde{B}_1 = B_1 + RB_0 - B_0R$$

is real and symmetric. Assuming that $R : \mathbf{X} \to \mathbf{X}$ is also real and symmetric we obtain

$$RB_0 - B_0R = \frac{1}{2}(B_1^* - B_1) = \frac{1}{2}[(PB)^2\, U^2BP - PBU^2\,(BP)^2],\ B_0 = PBP.$$

Then

$$B_0\left(R + \frac{1}{2}PBU^2BP\right) = \left(R + \frac{1}{2}PBU^2BP\right)B_0,$$

and this obviously leads to the simplest choice

$$R = -\frac{1}{2}PBU^2BP,$$

that satisfies our assumptions (R is real and symmetric). Thus, the following statement is proved.

Proposition 1. *The substitution*

$$z = x - \frac{\varepsilon^2}{2}PBU^2BPx + \dots,\ x = z + \frac{\varepsilon^2}{2}PBU^2BPz + \dots, \tag{66}$$

leads to "symmetric" Burnett equations

$$\begin{aligned} z_t + i\left[PBP + \frac{\varepsilon^2}{2}PB(U^2BP + PBU^2 - 2UBU)BP\right]z \\ +\varepsilon PBUBPz = 0,\ z \in \mathbf{X}, \end{aligned} \tag{67}$$

satisfying

$$\frac{1}{2}\frac{d}{dt}\,\|z\|^2 = -\varepsilon(Bz, UBz) \le 0. \tag{68}$$

Eq. (68) follows from the general identity (54) and semi-positivity of U (58). We can now use Eqs. (66) , (67) for the linearized Boltzmann equation (51) written in the form (52) and derive "symmetric" linearized Burnett equations. It is, however, more important to consider the nonlinear case. The same idea of substitutions (64) (with a nonlinear operator R) can be used for nonlinear equations of hydrodynamics. Then we hope to obtain a generalized version of nonlinear Burnett equations that corresponds to Eqs. (67) in the linearized case. We shall try to realize this program in Sections 6–7.

6. EQUATIONS OF HYDRODYNAMICS AND THEIR TRANSFORMATION

We return now to notations of Sections 2,3 and consider Eqs. (8). Our aim is to find a class of transformations $(\rho, u, T) \to (\rho', u', T')$ that can "regularize" the Burnett equations. If we assume that these transformations preserve a form of conservation laws (8), then it is easy to determine at least one of appropriate classes.

Proposition 2. *Let* $P_{\alpha\beta}(x,t) = P_{\beta\alpha}(x,t)$ *and* $Q_\alpha(x,t)$ $(\alpha, \beta = 1, 2, 3)$ *be arbitrary tensor and vector respectively. Then the substitution*

$$\rho' = \rho,\ \rho' u' = \rho u + \theta\varepsilon^2 w,\ \rho'(\left|u'\right|^2 + 3T') = \rho(|u|^2 + 3T) + 2\theta\varepsilon^2 s, \tag{69}$$

where θ *is a parameter (real number),*

$$w_\alpha = \frac{\partial}{\partial x_\beta} P_{\alpha\beta},\ s = \frac{\partial}{\partial x_\alpha}(u_\beta P_{\alpha\beta} + Q_\alpha),\ \alpha, \beta = 1, 2, 3, \tag{70}$$

transforms Eqs. (8) *to the following form (primes are omitted below):*

$$\rho_t + div\, \rho u = \theta\varepsilon^2 div\, w,$$

$$\frac{\partial}{\partial t}\rho u_\alpha + \frac{\partial}{\partial x_\beta}\left(\rho(u_\alpha u_\beta + T\delta_{\alpha\beta}) + \varepsilon\pi_{\alpha\beta}\right)$$

$$= \theta\varepsilon^2 \frac{\partial}{\partial x_\beta}\left[u_\alpha w_\beta + u_\beta w_\alpha + \frac{2}{3}U\delta_{\alpha\beta} + \frac{\partial P_{\alpha\beta}}{\partial t} + O(\varepsilon)\right],$$

$$\frac{1}{2}\frac{\partial}{\partial t}\rho\left(|u|^2 + 3T\right) + \frac{1}{2} div\, \rho u\left(|u|^2 + 5T\right) + \varepsilon\frac{\partial}{\partial x_\alpha}\left(\pi_{\alpha\beta}u_\beta + q_\alpha\right)$$

$$= \theta\varepsilon^2 \frac{\partial}{\partial x_\alpha}\left[w_\alpha \frac{|u|^2 + 5T}{2} + \left(s + \frac{2}{3}U\right)u_\alpha + \frac{\partial}{\partial t}(P_{\alpha\beta}u_\beta + Q_\alpha) + O(\varepsilon)\right],$$

$$U = s - u \cdot w = P_{\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta} + div\, Q \tag{71}$$

Remark 2. The notations $P_{\alpha\beta}$ and Q_α for arbitrary functions in Eqs. (70) should not be confused with notations of Eqs. (27),(28). We shall see below that the "correct" choice of $P_{\alpha\beta}$ and Q_α coincides with Eqs. (28) and this explains why we use the same notation.

The proof of Proposition 2 is straightforward and therefore we omit it. Note that the scalar $U(x,t)$ is directly related to the temperature T:

$$T' = T + \frac{2}{3}\frac{\theta\varepsilon^2}{\rho}U + O(\varepsilon^4) \tag{72}$$

We omit terms of order $O(\varepsilon^3)$ in Eqs. (71) and study the resulting equations. It is convenient to transform them to more explicit form. We denote

$$G_{\alpha\beta} = \mathcal{D}_0 P_{\alpha\beta} + P_{\alpha\beta}\, div\, u - P_{\alpha\gamma}\frac{\partial u_\beta}{\partial x_\gamma}, \mathcal{D}_0 = \frac{\partial}{\partial t} + u \cdot \frac{\partial}{\partial x},$$

$$S_\alpha = \mathcal{D}_0 Q_\alpha + Q_\alpha\, div\, u - Q_\beta \frac{\partial u_\alpha}{\partial x_\beta} + P_{\alpha\beta}\mathcal{D}_0 u_\beta + T w_\alpha, \alpha, \beta = 1, 2, 3. \quad (73)$$

Then it is easy to verify that

$$\frac{\partial}{\partial x_\beta}\left(G_{\alpha\beta} - \frac{\partial}{\partial t}P_{\alpha\beta} - u_\beta w_\alpha\right) = 0,$$

$$\frac{\partial}{\partial x_\alpha}\left[G_{\beta\alpha}u_\beta + S_\alpha - \frac{\partial}{\partial t}(P_{\alpha\beta}u_\beta + Q_\alpha) - s u_\alpha - T w_\alpha\right] = 0$$

where s and w are given in Eqs. (70). Therefore Eqs. (71) (without terms of order $O(\varepsilon^3)$) can be transformed to the form

$$\rho_t + div\,(\rho u - \theta\varepsilon^2 w) = 0,$$

$$\frac{\partial}{\partial t}\rho u_\alpha + \frac{\partial}{\partial x_\beta}\left[\rho(u_\beta - \theta\varepsilon^2 w_\beta)u_\alpha + (p - \frac{2}{3}\theta\varepsilon^2 U)\delta_{\alpha\beta} + \varepsilon\pi_{\alpha\beta} - \theta\varepsilon^2 G_{\alpha\beta}\right] = 0,$$

$$\frac{1}{2}\frac{\partial}{\partial t}\rho\left(|u|^2 + 3T\right) + \frac{\partial}{\partial x_\alpha}\left[\frac{1}{2}(\rho u_\alpha - \theta\varepsilon^2 w_\alpha)\left(|u|^2 + 3T\right)\right.$$

$$\left. + u_\alpha\left(p - \frac{2}{3}\theta\varepsilon^2 U\right) + \varepsilon(\pi_{\alpha\beta} - \theta\varepsilon G_{\beta\alpha})u_\beta + \varepsilon q_\alpha - \theta\varepsilon^2 S_\alpha\right] = 0, p = \rho T. \quad (74)$$

Finally we present the equations in the explicit (with respect to the time-derivatives) form:

$$\rho_t + div\, J = 0, J_\alpha = \rho u_\alpha - \theta\varepsilon^2\frac{\partial}{\partial x_\beta}P_{\alpha\beta},$$

$$\rho\frac{\partial u_\alpha}{\partial t} + J_\beta\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial\widetilde{p}}{\partial x_\alpha} + \varepsilon\frac{\partial\widetilde{\pi}_{\alpha\beta}}{\partial x_\beta} = 0,$$

$$\frac{3}{2}\left(\rho\frac{\partial T}{\partial t} + J_\beta\frac{\partial T}{\partial x_\beta}\right) + \widetilde{p}\, div\, u + \varepsilon\left(\widetilde{\pi}_{\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta} + div\,\widetilde{q}\right) = 0, \quad (75)$$

where

$$\widetilde{p} = p - \frac{2}{3}\theta\varepsilon^2 U, U = P_{\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta} + div\, Q,$$

$$\widetilde{\pi}_{\alpha\beta} = \pi_{\alpha\beta} - \theta\varepsilon G_{\alpha\beta}, \widetilde{q}_\alpha = q_\alpha - \theta\varepsilon S_\alpha, \alpha, \beta = 1, 2, 3. \quad (76)$$

We can also simplify Eqs. (73) since $\mathcal{D}_0 u = \rho^{-1}\nabla p + O(\varepsilon)$. Therefore

$$P_{\alpha\beta}\mathcal{D}_0 u_\beta + T w_\alpha = -\frac{1}{\rho} P_{\alpha\beta}\frac{\partial p}{\partial x_\beta} + T\frac{\partial}{\partial x_\beta} P_{\alpha\beta} + O(\varepsilon) = \rho T^2 \frac{\partial}{\partial x_\beta}\frac{P_{\alpha\beta}}{\rho T} + O(\varepsilon).$$

Hence, $G_{\alpha\beta}$ and S_α in Eqs. (76) can be calculated (with the same accuracy) by formulas

$$G_{\alpha\beta} = \mathcal{D}_0 P_{\alpha\beta} + P_{\alpha\beta}\, div\, u - P_{\alpha\gamma}\frac{\partial u_\beta}{\partial x_\gamma},$$

$$S_\alpha = \mathcal{D}_0 Q_\alpha + Q_\alpha\, div\, u - Q_\beta \frac{\partial u_\alpha}{\partial x_\beta} + \rho T^2 \frac{\partial}{\partial x_\beta}\frac{P_{\alpha\beta}}{\rho T}. \tag{77}$$

Thus, Eqs. (75) – (77) define uniquely a set of equations of hydrodynamics in new variables (69) provided (i) $\pi_{\alpha\beta}$ and q_α in Eqs. (8) are known and (ii) arbitrary functions $P_{\alpha\beta}(x,t)$, $Q_\alpha(x,t)$ (70) and the parameter θ are fixed. It is obvious that this transformation makes sense if and only if $\pi_{\alpha\beta}$ and q_α are evaluated at the Burnett level (see Eqs. (23)). All considerations, however, remain the same if we consider higher approximations for $\pi_{\alpha\beta}$ and q_α. In such cases we just need to choose ε^n, $n > 2$, instead of ε^2 in Eqs. (70) and this again leads to equations similar to Eqs. (75). The same is true for $n = 1$ (generalized Navier-Stokes equations).

We shall see below that this transformation leads to well-posed (hyperbolic) Burnett equations.

7. HYPERBOLICITY AND LINEARIZED H-THEOREM

We assume now that $\pi_{\alpha\beta}$ and q_α in Eqs. (76) are given in Eqs. (23) , (27) (see also more explicit formulas (33) and (40) , (41) for Navier-Stokes terms). Then we choose $P_{\alpha\beta}(x,t)$ and $Q_\alpha(x,t)$ in Eqs. (70) , (75) , (76) in the form given in Eqs. (29), i.e.

$$P_{\alpha\beta} = \frac{A}{\rho}\overline{\frac{\partial u_\alpha}{\partial x_\beta}},\; Q_\alpha = \frac{B}{\rho}\frac{\partial T}{\partial x_\alpha},$$

where $A(T)$ and $B(T)$ are given in Eqs. (30). The terms $\mathcal{D}_0 P_{\alpha\beta}$ and $\mathcal{D}_0 Q_\alpha$ in Eqs. (77) should be evaluated at the "Euler approximation." This was already done in Section 3, see Eqs. (32). Hence, all terms in Eqs. (75) – (77) are uniquely determined, except for the parameter θ. Then we can consider Eqs. (75) – (77) as a one-parameter family of generalized Burnett equations. In order to find an appropriate value of θ we repeat for Eqs. (75) the same considerations as for classical Burnett equations (the end of Section 3). Keeping just terms with higher derivatives we obtain

$$\rho_t = \theta\varepsilon^2 \frac{\partial^2}{\partial x_\alpha \partial x_\beta} \frac{A}{\rho} \overline{\frac{\partial u_\alpha}{\partial x_\beta}} + \ldots = \frac{2}{3}\theta\varepsilon^2 \frac{A}{\rho} \triangle\, div\, u + \ldots ,$$

$$\rho \frac{\partial u_\alpha}{\partial t} = -\varepsilon^2 \frac{\partial}{\partial x_\beta} \pi^B_{\alpha\beta} + \theta\varepsilon^2 \left(\frac{2}{3} \frac{\partial U}{\partial x_\alpha} + \frac{\partial G_{\alpha\beta}}{\partial x_\beta} \right) + \ldots$$

$$= -\varepsilon^2 \frac{\partial}{\partial x_\beta} \pi^B_{\alpha\beta} + \frac{2\theta\varepsilon^2}{3\rho} \frac{\partial}{\partial x_\alpha} \left(B \triangle T - \frac{A}{\rho} \triangle \rho \right) + \ldots ,$$

$$\frac{3}{2}\rho T_t = -\varepsilon^2 div\, q^B + \theta\varepsilon^2 div\, S + \ldots$$

$$= -\varepsilon^2 div\, q^B + \frac{2\theta\varepsilon^2}{3\rho} T(A - B) \triangle\, div\, u + \ldots , \tag{78}$$

The contribution of Burnett terms π^B and q^B was already found (see Eqs. (33)). Then, we consider again $1d$ solutions and obtain, omitting terms with lower derivatives,

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ u \\ T \end{pmatrix} = \frac{2}{3} \frac{\varepsilon^2}{\rho} M(\theta) \left(\frac{\partial}{\partial x} \right)^3 \begin{pmatrix} \rho \\ u \\ T \end{pmatrix} + \ldots , x \in \mathbb{R}, u \in \mathbb{R}.$$

The matrix $M(\theta)$ reads

$$M(\theta) = \begin{pmatrix} 0 & 0 & 0 \\ \frac{AT}{\rho} & 0 & A - C \\ 0 & \frac{2(B-C)T}{3} & 0 \end{pmatrix} + \theta \begin{pmatrix} 0 & \rho A & 0 \\ -\frac{AT}{\rho} & 0 & B - A \\ 0 & \frac{2(A-B)T}{3} & 0 \end{pmatrix} ,$$

where the first term $M(0)$ corresponds to usual Burnett equations (see. Eq. (37)). One can easily guess (by analogy with linear theory of Sections 4,5) that the best result is achieved for $\theta = 1/2$. In such a case we obtain

$$M(1/2) = \frac{1}{2} \begin{pmatrix} 0 & \rho A & 0 \\ \frac{AT}{\rho} & 0 & A + B - 2C \\ 0 & \frac{2T}{3}(A + B - 2C) & 0 \end{pmatrix} .$$

The eigenvalues of $M(1/2)$ are obviously real

$$\lambda = 0, \lambda = \pm \frac{2T^{1/2}}{2} \left[\frac{2}{3}(A + B - 2C)^2 + A^2 \right]^{1/2}$$

provided $T > 0$. Thus the generalized Burnett equations are hyperbolic for any intermolecular potential (that defines coefficients $A(T)$, $B(T)$, $C(T)$, see Eqs. (30)) if $\theta = 1/2$ and $T > 0$. Therefore we fix the value $\theta = 1/2$ and call the corresponding Eqs. (75) – (78) “Hyperbolic Burnett equations” (HBEs).

Let us check that HBEs guarantee the stability of any constant (equilibrium) solution $\rho = \rho_0 > 0$, $T = T_0 > 0$, $u = u_0 = 0$ (the equations are obviously Galilei-invariant, therefore we can always choose $u_0 = 0$). We denote

$$\rho = \rho_0[1 + \rho'(x, t')], u = T_0^{1/2} u'(x, t'), T = T_0[1 + T'(x, t')], t' = T_0^{1/2} t$$

and consider linearized HBEs. Then the terms with third derivatives can be taken from Eqs. (78), (33). The terms with second derivatives appear due to Navier-Stokes terms (40), (41). We obtain after simple calculations the following set of linear equations (primes are omitted):

$$\begin{aligned} &\rho_t + div\, u = a \,\triangle\, div\, u, \\ &u_t + \nabla\rho + \nabla T = c\left(\triangle u + \frac{1}{3}\nabla div\, u\right) + a\nabla(\triangle\rho) + b\nabla(\triangle T), \\ &\frac{3}{2}T_t + div\, u = d \,\triangle\, T + b \,\triangle\, div\, u, \end{aligned} \tag{79}$$

where

$$a = \varepsilon^2 \frac{A(T_0)}{3\rho_0^2}, b = \frac{\varepsilon^2}{3\rho_0^2}[A(T_0) + B(T_0) - 2C(T_0)], c = \varepsilon \frac{\mu(T_0)}{\rho_0 T_0^{1/2}}, d = \varepsilon \frac{\lambda(T_0)}{\rho_0 T_0^{1/2}}, \tag{80}$$

in the notations of Eqs. (30), (41).

We introduce "H-function"

$$H(x, t) = \frac{1}{2}\left(\rho^2 + |u|^2 + \frac{3}{2}T^2\right), \tag{81}$$

then

$$\begin{aligned} \frac{\partial H}{\partial t} + div\,[(\rho + T)u] = cu \cdot (\triangle u + \frac{1}{3}\nabla div\, u) + dT \,\triangle\, T + \\ + a[\rho \,\triangle\, div\, u + u \cdot \nabla(\triangle\rho)] + b[T \,\triangle\, div\, u + u \cdot \nabla(\triangle T)]. \end{aligned}$$

The identity

$$g \,\triangle\, div\, u + u \cdot \nabla(\triangle g) = div\, \Psi(u, g), \Psi(u, g) = \triangle g + g(\nabla div\, u) - (\nabla g) div\, u,$$

holds for any (smooth) scalar $g(x)$ and vector $u(x)$. Therefore we obtain

$$\frac{\partial H}{\partial t} + div\,[(\rho + T)u - \Psi(u, a\rho + bT)] = cu \cdot \left(\triangle u + \frac{1}{3}\nabla div\, u\right) + dT \,\triangle\, T.$$

Assuming that $\rho(x,t)$, $u(x,t)$ and $T(x,t)$ decay fast enough if $|x| \to \infty$, we obtain "H-theorem"

$$\frac{d}{dt}\int_{\mathbb{R}^3} dx\, H(x,t) = -c\int_{\mathbb{R}^3} dx\left[\left(\frac{\partial u_\alpha}{\partial x_\beta}\right)^2 + \frac{1}{3}(div\, u)^2\right] - d\int_{\mathbb{R}^3} dx\, |\nabla T|^2 \le 0, \tag{82}$$

where the summation over $\alpha, \beta = 1, 2, 3$ is assumed. The inequality follows from obvious positivity of c and d (see Eqs. (80), (41)). This proves that any constant solution of HBEs (with positive ρ_0 and T_0) is stable for any intermolecular potential.

8. HYPERBOLIC BURNETT EQUATIONS

We explain below the meaning and the structure of HBEs, assuming that all terms in the classical Burnett equations are known. HBEs are equations for *auxiliary* variables for which we keep the initial notations $\rho(x,t)$, $u(x,t)$ and $T(x,t)$. The *true* hydrodynamical variables $(\rho^{tr}, u^{tr}, T^{tr})$ are expressed through (ρ, u, T) by equalities (with accuracy $O(\varepsilon^2)$)

$$\rho^{tr} = \rho,\, u_\alpha^{tr} = \widehat{u}_\alpha(\rho, u, T) = u_\alpha - \frac{\varepsilon^2}{2\rho}\frac{\partial}{\partial x_\beta}\frac{A}{\rho}\overline{\frac{\partial u_\alpha}{\partial x_\beta}},$$

$$T^{tr} = \widehat{T}(\rho, u, T) = T - \frac{\varepsilon^2}{3\rho}\left(\frac{A}{\rho}\overline{\frac{\partial u_\alpha}{\partial x_\beta}}\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial}{\partial x_\alpha}\frac{B}{\rho}\frac{\partial T}{\partial x_\alpha}\right), \alpha, \beta = 1, 2, 3, \tag{83}$$

where $A(T)$ and $B(T)$ are the Burnett coefficients defined in Eqs. (30).

The equations for (ρ, u, T) read

$$\rho_t + div\,\rho\widehat{u} = 0,\, \rho\left(\frac{\partial}{\partial t} + \widehat{u}\cdot\frac{\partial}{\partial x}\right)u_\alpha + \frac{\partial \Pi_{\alpha\beta}}{\partial x_\beta} = 0,$$

$$\frac{3}{2}\rho\left(\frac{\partial}{\partial t} + \widehat{u}\cdot\frac{\partial}{\partial x}\right)T + \Pi_{\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta} + \varepsilon div\, g = 0, \tag{84}$$

or, equivalently,

$$\rho_t + div\,\rho\widehat{u} = 0,\, \frac{\partial}{\partial t}\rho u_\alpha + \frac{\partial}{\partial x_\beta}(\rho u_\alpha \widehat{u}_\beta + \Pi_{\alpha\beta}) = 0,$$

$$\frac{\partial}{\partial t}\rho(|u|^2 + 3T) + \frac{\partial}{\partial x_\alpha}[\rho\widehat{u}_\alpha(|u|^2 + 3T) + 2(\Pi_{\beta\alpha}u_\beta + \varepsilon g_\alpha)] = 0, \tag{85}$$

where

$$\Pi_{\alpha\beta} = \rho\widehat{T}\delta_{\alpha\beta} - 2\varepsilon\mu\overline{\frac{\partial u_\alpha}{\partial x_\beta}} + \varepsilon^2\left(\pi^B_{\alpha\beta} - \frac{1}{2}G_{\alpha\beta}\right),$$

$$g_\alpha = -\lambda\frac{\partial T}{\partial x_\alpha} + \varepsilon\left(q^B_\alpha - \frac{1}{2}S_\alpha\right), \alpha, \beta = 1, 2, 3. \tag{86}$$

The notations $\mu = \mu(T)$, $\lambda = \lambda(T)$ and $\pi^B_{\alpha\beta}$, q^B_α correspond respectively to the Navier-Stokes coefficients and the Burnett terms that were discussed in detail in Section 3. The additional terms $G_{\alpha\beta}$ and S_α were defined in Eqs. (77) (note that $G_{\alpha\beta} \neq G_{\beta\alpha}$ and therefore $\Pi_{\alpha\beta} \neq \Pi_{\beta\alpha}$!). We present below more explicit formulas for these terms:

$$G_{\alpha\beta} = G^{(0)}_{\alpha\beta} + \frac{A}{\rho}\left(\overline{\frac{\partial u_\alpha}{\partial x_\beta}}div\, u - \overline{\frac{\partial u_\alpha}{\partial x_\gamma}\frac{\partial u_\beta}{\partial x_\gamma}}\right),$$

$$S_\alpha = S^{(0)}_\alpha + \frac{B}{\rho}\left(\frac{\partial T}{\partial x_\alpha}div\, u - \frac{\partial T}{\partial x_\beta}\frac{\partial u_\alpha}{\partial x_\beta}\right) + \rho T^2\frac{\partial}{\partial x_\beta}\frac{A}{\rho^2 T}\overline{\frac{\partial u_\alpha}{\partial x_\beta}}, \tag{87}$$

where

$$G^{(0)}_{\alpha\beta} = \mathcal{D}_0 P_{\alpha\beta} = -\frac{A}{\rho}\left(\overline{\frac{\partial}{\partial x_\alpha}\frac{1}{\rho}\frac{\partial p}{\partial x_\beta}} + \overline{\frac{\partial u_\alpha}{\partial x_\gamma}\frac{\partial u_\gamma}{\partial x_\beta}}\right) + \frac{1}{\rho}\left(A - \frac{2}{3}T\frac{dA}{dT}\right)\overline{\frac{\partial u_\alpha}{\partial x_\beta}}div\, u,$$

$$S^{(0)}_\alpha = \mathcal{D}_0 Q_\alpha = -\frac{B}{\rho}\left(\frac{2}{3}\frac{\partial}{\partial x_\alpha}T div\, u + \frac{\partial u_\beta}{\partial x_\alpha}\frac{\partial T}{\partial x_\beta}\right) + \frac{1}{\rho}\left(B - \frac{2}{3}T\frac{dB}{dT}\right)\frac{\partial T}{\partial x_\alpha}div\, u,$$

$$p = \rho T, \alpha, \beta, \gamma = 1, 2, 3. \tag{88}$$

The reader might think that the additional terms make HBEs much more sophisticated as compared to the classical Burnett equations. Fortunately this is not true, since both $G_{\alpha\beta}$ and $\pi^B_{\alpha\beta}$ (S_α and q^B_α respectively) are, roughly speaking, linear combinations of similar terms. We remind to the reader (see Section 3) that

$$\pi^B_{\alpha\beta} = G^{(0)}_{\alpha\beta} + \frac{1}{\rho}\overline{\frac{\partial}{\partial x_\alpha}C\frac{\partial T}{\partial x_\beta}} - \frac{\Delta_{\alpha\beta}}{\rho},$$

$$q^B_\alpha = S^{(0)}_\alpha + \frac{1}{\rho}\frac{\partial}{\partial x_\beta}CT\overline{\frac{\partial u_\alpha}{\partial x_\beta}} - \frac{\Delta_\alpha}{\rho}, \alpha, \beta = 1, 2, 3, \tag{89}$$

where $C(T)$, $\Delta_{\alpha\beta}$, Δ_α are defined in Eqs. (28), (30).

It should be stressed that HBEs (84) – (89) are hyperbolic and satisfy the linearized H-theorem (82) for any choice of coefficients $A(T)$, $B(T)$, $C(T)$ provided $T > 0$. These properties do not depend on $\Delta_{\alpha\beta}$ and Δ_α in Eqs. (89). All above terms can be computed exactly in the case of Maxwell molecules that corresponds to the cross-section $\sigma(|u|, \cos\theta) = |u|^{-1} g(\cos\theta)$ in Eq. (2). Then obtain after some calculations

$$\mu(T) = \eta T, \lambda(T) = \frac{15}{4}\eta T, \eta^{-1} = \frac{3\pi}{2}\int_{-1}^{1} d\mu\, g(\mu)(1-\mu^2),$$

$$\varphi_{\alpha\beta} = -\eta\overline{c_\alpha c_\beta}, \varphi_\alpha = -\frac{3}{4}\eta c_\alpha(|c|^2 - 5T), A(T) = 2\eta^2 T, B(T) = \frac{45}{8}\eta^2 T,$$

$$C(T) = 3\eta^2 T, \Delta_{\alpha\beta} = \langle F_0, \mathcal{D}\varphi_{\alpha\beta}\rangle = -4\eta^2 T\overline{\frac{\partial u_\alpha}{\partial x_\gamma}\frac{\partial u_\beta}{\partial x_\gamma}},$$

$$\Delta_\alpha = \langle F_0, \mathcal{D}\varphi_\alpha\rangle = -3\eta^2 T\left[\left(4\frac{\partial T}{\partial x_\beta} - \frac{1}{\rho}\frac{\partial p}{\partial x_\beta}\right)\overline{\frac{\partial u_\alpha}{\partial x_\beta}} + \frac{15}{8}\frac{\partial T}{\partial x_\beta}\frac{\partial u_\alpha}{\partial x_\beta} + \frac{5}{4}\frac{\partial T}{\partial x_\alpha} div\, u\right]. \quad (90)$$

In the general case we can use the standard approximation [8]

$$\lambda(T) = \frac{15}{4}\mu(T), \varphi_{\alpha\beta} = -\frac{\mu}{T}\overline{c_\alpha c_\beta}, \varphi_\alpha = -\frac{3\mu}{4}c_\alpha(|c|^2 - 5T), \quad (91)$$

that leads (see Eqs. (45)) to equalities

$$A = 2\frac{\mu^2}{T}, B = \frac{45}{8}\frac{\mu^2}{T}, C = 3\frac{\mu^2}{T}. \quad (92)$$

The corresponding approximate formulas for $\pi^B_{\alpha\beta}$ and q^B_α are well-known [8, 16] (see also [21]). These formulas, combined with Eqs. (84) – (89), (91), define approximate HBEs for general intermolecular potential.

The most general formulas for $\pi^B_{\alpha\beta}$ and q^B_α (without the assumption (91)) express these quantities through 11 coefficients $\{\omega_1(T), \dots, \omega_6(T); \theta_1(T), \dots, \theta_5(T)\}$ that depend on intermolecular potential [10]. It was recently proved by Slemrod that $\omega_3 + \omega_4 + \theta_3 = 0$ [21]. This identity becomes almost obvious if we use the notations of the present paper. Then it is easy to verify by comparison with [10] that

$$\omega_3 = \frac{1}{5T}\langle\varphi_{\alpha\beta}, \varphi_\alpha c_\beta\rangle_M = TC(T), \omega_4 = -\frac{1}{5}\left\langle\varphi_\alpha, \frac{\partial\varphi_{\alpha\beta}}{\partial c_\beta}\right\rangle_M,$$

$$\theta_3 = -\frac{1}{5}\left\langle\varphi_{\alpha\beta}, \frac{\partial\varphi_\alpha}{\partial c_\beta}\right\rangle_M, \quad (93)$$

where the functions $\varphi_\alpha(c;T), \varphi_{\alpha\beta}(c;T)$ $(\alpha, \beta = 1, 2, 3)$ are given in Eqs. (19). The Slemrod identity follows from Eqs. (93) after integration by parts.

APPENDIX. A. COMPLETE CHAPMAN-ENSKOG EXPANSION FOR LINEAR EQUATIONS

This part of the paper can be considered as a continuation of Section 4, the same notations are used below. We consider the Cauchy problem (52)

$$\varepsilon u_t + L(i\varepsilon)u = 0,\ u|_{t=0} = u_0; L(\kappa) = A + \kappa B, \tag{A1}$$

under the assumptions [i]–[iii]. Our goal is to understand the general structure of the Chapman-Enskog expansion and to show why the change of coordinates (hydrodynamical variables for the linearized Boltzmann equation (49)) seems to be very natural for this problem.

Let us consider the operator $L(\kappa) : \mathbf{E} \to \mathbf{E}$ for small complex κ, $|\kappa| < r_0$. First we assume that $\dim \mathbf{E} < \infty$. Then, for any real $\kappa \in \mathbb{R}$, the operator $L(\kappa)$ is symmetric and therefore

$$L(\kappa) = \sum_{j=1}^{s} \lambda_j(\kappa) Q_j(\kappa), s = \dim \mathbf{E}, \tag{A2}$$

where $\lambda_j(\kappa)$ are eigenvalues and $Q_j(\kappa)$ are corresponding one-dimensional proper projectors of $L(\kappa)$. We assume below that $\ker L(\kappa) = \emptyset$ if $\kappa \neq 0, |\kappa| < r_0$. It is well known that that $\lambda_j(\kappa)$ and $Q_j(\kappa)$ are analytic functions of κ in some neighborhood $|\kappa| < r_0$ of $\kappa = 0$ [15]. Then, by analytic continuation, Eq. ($A2$) holds for small complex κ, though the projectors $Q_j(\kappa)$ are orthogonal ($Q_j^* = Q_j$) only for real κ. The identity (54) implies that $\mathrm{Re}\lambda_j(i\varepsilon) \geq 0$ for $\varepsilon > 0$. We can denumerate the eigenvalues in such a way that

$$\lambda_1(0) = \ldots = \lambda_m(0) = 0; \lambda_j(0) > 0, m + 1 \leq j \leq s,$$

where $\lambda_j(0)$, $j \geq m + 1$, are positive eigenvalues of A. Then the solution of the problem ($A1$) reads

$$u(t;\varepsilon) = \exp\left[-\frac{t}{\varepsilon} L(i\varepsilon)\right] u_0 = w(t;\varepsilon) + O(e^{-\frac{ct}{\varepsilon}}), \tag{A3}$$

where c is a positive constant, $t > 0$, $\varepsilon \to 0^+$,

$$w(t;\varepsilon) = \sum_{k=1}^{m} e^{-\frac{t}{\varepsilon}\lambda_k(i\varepsilon)} Q_k(i\varepsilon) u_0,\ \lambda_k(\kappa) = O(\kappa), k = 1, \ldots, m. \tag{A4}$$

Following Grad [12], we call $w(t;\varepsilon)$ the normal solution of the problem ($A1$).

Remark 3: The simplified assumption $\dim \mathbf{E} < \infty$ leads immediately to the estimate ($A3$). The same estimate, however, can be obtained in many cases when $\dim \mathbf{E} = \infty$. In particular, such estimate for the linearized Boltzmann equation (hard sphere model) was proved long ago by Arsen'ev [2].

Hence, in order to study asymptotics for small positive ε it is sufficient to consider the normal solution $w(t;\varepsilon)$. We note that

$$w(t;\varepsilon) = \Pi(i\varepsilon)u(t;\varepsilon),\ \Pi(\kappa) = \sum_{j=1}^{m} Q_j(\kappa),$$

where $\Pi(\kappa)$ is an m-dimensional proper projector of $L(\kappa)$ obtained by perturbation of the orthogonal projector $P = \Pi(0)$ onto the subspace $\mathbf{X} = \mathbf{N}(A)$ (see Eqs. (55) , (56)). Fortunately the general formulas for $\Pi(\kappa)$ are known [15]. We just need to apply them to the particular case of $L(\kappa)$ $(A1)$. Omitting details of calculations, we obtain

$$\Pi(\kappa) = \sum_{n=0}^{\infty} \kappa^n Q^{(n)},\ Q^{(0)} = P,\ Q^{(n)} = Res\left\{\frac{1}{z^{n+1}} C(z)[BC(z)]^n\right\},\ n = 1,\ \dots, \tag{A5}$$

where

$$Res\ F(z) = \lim_{r\to 0^+} \frac{1}{2\pi i} \oint_{|z|=r} F(z)\,dz,$$

$$C(z) = P - \sum_{n=1}^{\infty} z^n U^n,\ U = (1-P)A^{-1}(1-P) \tag{A6}$$

(note that U is the same as in Eq. (58) since $(1-P)A^{-1} = A^{-1}(1-P)$, $(1-P)^2 = (1-P)$). The calculation of residues in Eqs. $(A5)$ is very simple since all operator-valued functions are represented by series

$$F(z) = \sum_{k=-n}^{\infty} F_k z^k,\ n = 2, 3,\ \dots, \tag{A7}$$

then $Res\ F(z) = F_{-1}$.

Now we can construct the complete Chapman-Enskog expansion. We fix a small $\varepsilon > 0$ and simplify notations by denoting

$$Q = \Pi(i\varepsilon) = \sum_{n=0}^{\infty} (i\varepsilon)^n Q^{(n)},\ L = L(i\varepsilon),\ u = u(t;\varepsilon),\ w = w(t;\varepsilon), \tag{A8}$$

and so on. Then the vector

$$x(t;\varepsilon) = x = Pw = PQu \tag{A9}$$

describes for $\varepsilon \to 0$ the "hydrodynamical" part of the solution. The equation for x reads

$$\varepsilon x_t + PQLu = 0,\ x|_{t=0} = PQu_0. \tag{A10}$$

In order to close this equation we note that the projector Q was constructed in such a way that $QL = LQ$. Hence, it is sufficient to express $w = Qu$ through x. This can be done in the following way.

We consider the operator identities [15] that hold for any pair of projectors P and Q (i.e. operators such that $P^2 = P$, $Q^2 = Q$):

$$PT = TP = PQP,\, QT = TQ = QPQ,\, T = 1 - (P - Q)^2.$$

If $\|P - Q\| < 1$, then T is invertible and

$$Q = T^{-1}QPQ = (QP)T^{-1}(PQ).$$

By applying this identity to Eq. $(A10)$ we obtain

$$\varepsilon x_t = -PQLu = -PQLQu = -(PQLQP)T^{-1}(PQu). \tag{A11}$$

On the other hand, $x = PQu$ (see Eq. $(A9)$) and we obtain

$$\varepsilon x_t + (PQLQP)T^{-1}x = 0,\; x|_{t=0} = PQu_0. \tag{A12}$$

We note that $L = A + i\varepsilon B$, $QL = LQ$, $PA = AP = 0$, therefore

$$PQLQP = i\varepsilon PQBP = i\varepsilon PBQP. \tag{A13}$$

Let us consider the vector $x' = T^{-1}x$. Then $Px' = x'$ since $Px = x$ and $PT = TP = PQP$. Hence,

$$x = Tx' = TPx' = PQPx'. \tag{A14}$$

By using Eqs. $(A13)$, $(A14)$ we transform Eq. $(A12)$ to the following form

$$x_t + i\Lambda x' = 0,\, x = Gx',\, G = PQP,\, \Lambda = PBQP. \tag{A15}$$

Note that both operators Λ and G act from $\mathbf{X}$ to $\mathbf{X}$, $\mathbf{X} = \mathbf{N}(A)$, and are represented by power series in ε through the projector $Q = \Pi(i\varepsilon)$ $(A8)$. The condition $\|P - Q\| < 1$ is satisfied for $\varepsilon \to 0$ and therefore there exist a unique operator $G^{-1} : \mathbf{X} \to \mathbf{X}$. Hence, the equation for $x = x(t;\varepsilon)$ reads

$$x_t + i\Lambda G^{-1}x = 0,\; x|_{t=0} = PQu_0. \tag{A16}$$

Explicit formulas for Λ and G can be easily obtained from Eqs. $(A5)$, $(A6)$, $(A15)$. We note that $PC(z) = C(z)P = P$, therefore

$$\Lambda = P\left(\sum_{n=0}^{\infty}(i\varepsilon)^n\Lambda_n\right)P,\, G = P\left(\sum_{n=0}^{\infty}(i\varepsilon)^n G_n\right)P, \tag{A17}$$

where

$$\Lambda_0 = B,\, \Lambda_1 = -BUB,\, \Lambda_n = Res\left[\frac{B[C(z)B]^n}{z^{n+1}}\right],$$

$$G_0 = 1,\, G_1 = 0,\, G_n = Res\left[\frac{B[C(z)B]^{n-1}}{z^{n+1}}\right],\, n = 2, 3, \,\dots, \tag{A18}$$

all residues are calculated by using the Laurent series ($A7$).

It is clear that the representation in the form of power series (in ε) is unique. Therefore we obtain the following result.

Proposition 3. *The complete Chapman-Enskog expansion for Eq.* ($A1$) *leads to Eqs.* ($A16$)*, where the operator* $G^{-1} : \mathbf{X} \to \mathbf{X}$ *is obtained by inversion of the power series* ($A17$) *for* G*, i.e.*

$$G^{-1} = P \left\{ 1 + \varepsilon^2 Res \left[\frac{BC(z)B}{z^3} \right] + \dots \right\} P. \tag{A19}$$

The correct initial conditions are also given in Eqs. ($A16$)*. The power series for* Λ *and* G^{-1} *are convergent for* $|\varepsilon| < r_0$ *with sufficiently small* $r_0 > 0$ *provided* $\dim \mathbf{E} < \infty$.

The latter statement follows from general results of the perturbation theory in finite-dimensional spaces [15]. Probably a similar statement can be proved for the Fourier-transformed Boltzmann equation (51) (hard sphere model) with sufficiently small $|k|$. We, however, do not consider a problem of convergence in this paper.

Thus, we know the general form of "equations of hydrodynamics" ($A16$). The operators Λ and G can be represented as

$$\Lambda = \Lambda^{(0)}(\varepsilon^2) + i\varepsilon\Lambda^{(1)}(\varepsilon^2), G = G^{(0)}(\varepsilon^2) + i\varepsilon G^{(1)}(\varepsilon^2), \tag{A20}$$

where both $\Lambda^{(0,1)}$, $G^{(0,1)}$ are real symmetric operators, such that

$$\Lambda^{(0)} = PBP - \varepsilon^2 PB(UBU - U^2BP - PBU^2)BP + O(\varepsilon^4),$$

$$G^{(0)} = P + \varepsilon^2 PBU^2BP + O(\varepsilon^4), \Lambda^{(1)} = -PBUBP + O(\varepsilon^2), G^{(1)} = O(\varepsilon^2).$$

This leads precisely to Eqs. (60). Thus, the loss of symmetry (discussed at the end of Section 4) is caused by a special structure of the operator ΛG^{-1} (a product of two "quasi-symmetric" operators of the form ($A20$)). Roughly speaking, the operator Λ is responsible for the dynamics, whereas the operator $G = PQP$ is related to the choice of coordinates. A natural "symmetrization" of the operator ΛG^{-1} can be obtained by substitution

$$x = G^{1/2}z, z \in \mathbf{X} \tag{A21}$$

(the vector $z \in \mathbf{X}$ should not be confused with the traditional notation for complex variable in Eqs. ($A18$) , ($A19$)). Then we obtain the equation

$$z_t + iHz = 0, H = G^{-1/2}\Lambda G^{-1/2}, z|_{t=0} = G^{-1/2}PQu_0,$$

where the operator $G^{-1/2} : \mathbf{X} \to \mathbf{X}$ is defined in the usual way

$$G^{-1/2} = P\left[1 - \frac{\varepsilon^2}{2} BU^2 B + O(\varepsilon^3)\right] P.$$

Then

$$H = H_0(\varepsilon^2) + i\varepsilon H_1(\varepsilon^2),$$

where both H_0 and H_0 are real symmetric operators. Therefore (see Eq. (54))

$$\frac{1}{2}\frac{d}{dt}\,\|z\|^2 = \varepsilon(H_1(\varepsilon^2)z, z)$$

i.e. $H_0(\varepsilon^2)$ does not influence this equality. The substitution (66) can be understood now as the first step of the general symmetrizing substitution ($A21$).

ACKNOWLEDGMENTS

The paper is dedicated to Carlo Cercignani. I am grateful to Carlo Cercignani, Leif Arkeryd, Marshall Slemrod and David Levermore for many stimulating discussions on the subject of this paper. Its results were presented in my talk at the HYKE conference in Rome (April 15–17, 2005). I thank Mario Pulvirenti for the invitation to the conference. I also thank participants of the conference for their questions and comments. I am grateful to my PhD students Niclas Bernhoff and Mirela Vinerean for their help. The research was performed in the frame of European IHP network HYKE (HPRN-2002-00282). The support from the Swedish Research Council (grant 621-2003-5357) is also gratefully acknowledged.

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